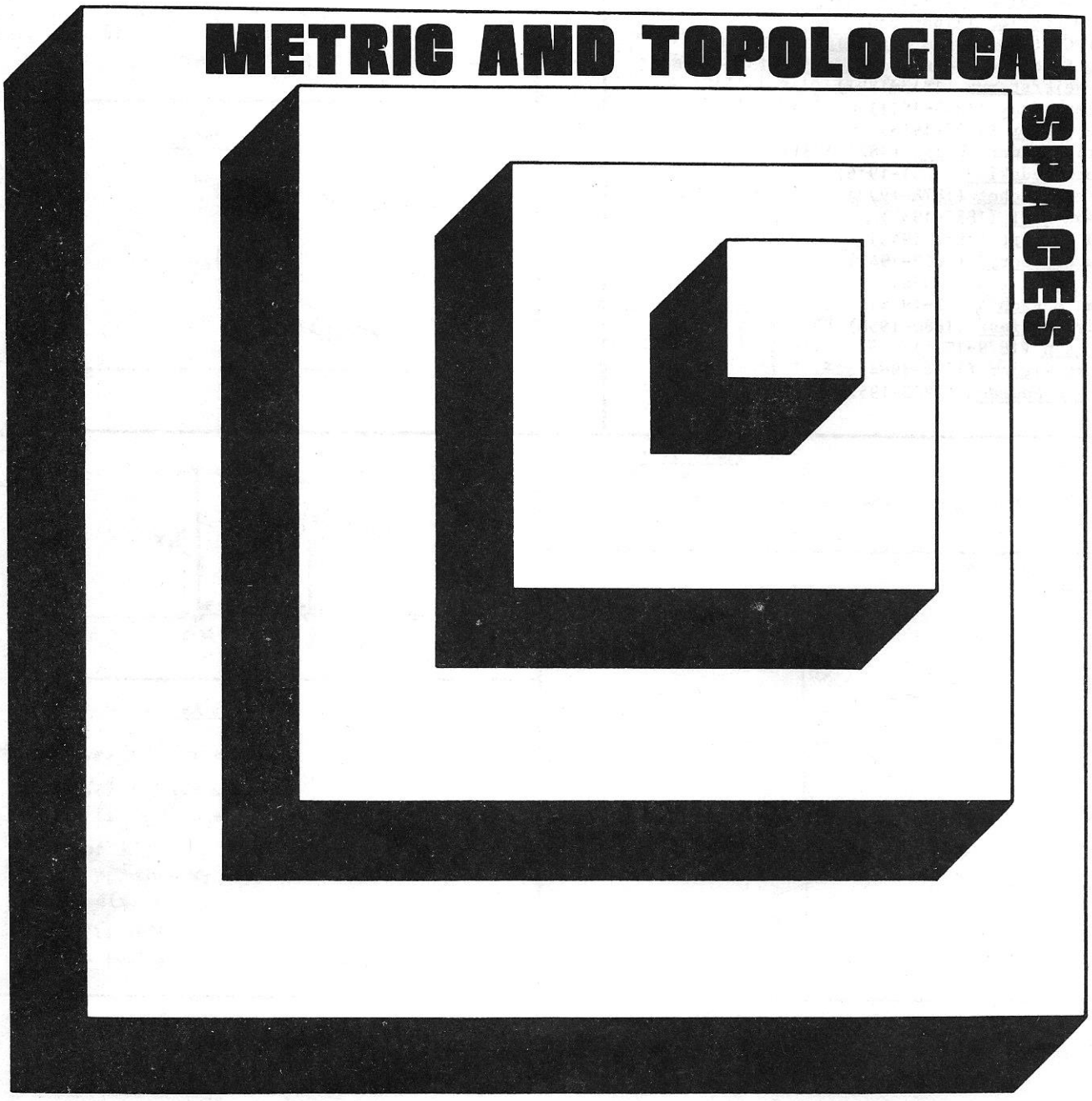
THE OPEN UNIVERSITY
Mathematics: A Third Level Course

M335
Tape Booklet



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The Historical Introduction

Tape 1 Side 1

FRAME 1

Mathematicians named in the tape - in the order in which they are named:
Johann Listing (1808-1882)
Carl Friedrich Gauss (1777-1855)
Henri Poincaré (1854-1912)
Brook Taylor (1685-1731)
James Gregory (1638-1675)
Isaac Newton (1642-1727)
Gottfried Wilhelm Leibniz (1646- 1716)
Leonhard Euler (1707-1783)
Jean Le Rond d'Álembert (1717-1783)
Joseph Louis Lagrange (1736-1813)
Augustin-Louis Cauchy (1789-1857)
Daniel Bernoulli (1700-1782)
Bernhard Bolzano (1781-1848)
Joseph Fourier (1768-1830)
Peter Gustav Lejeune Dirichlet (1805-1859)
Georg Friedrich Bernhard Riemann (1826-1866)
Karl Weierstrass (1815-1897)
Charles Méray (1835-1911)
Georg Cantor (1845-1918)
Heinrich Eduard Heine (1821-1881)
Richard Dedekind (1831-1916)
Maurice Fréchet (1878-1973)
Hermann Weyl (1885-1955)
David Hilbert (1862-1943)
Felix Hausdorff (1868-1942)
Emile Borel (1871-1956)
Stefan Banach (1892-1945)
Friedrich Riesz (1880-1956)
Hans Hahn (1879-1934)
Norbert Wiener (1894-1964)
John von Neumann (1903-1957)

FRAME 2

The trigonometric series referred to are those of the form:

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

An Example of a Metric

(Unit 3) Tape 1 Side 2

FRAME 1

A metric on a non-empty set A is a function $d : A \times A \to \mathbb{R}$ which satisfies:

(M1) $d(x,y) \geq 0$, for all x,y in A; $d(x,y) = 0$ if and only if $x = y$;

(M2) $d(x,y) = d(y,x)$, for all x,y in A;

(M3) $d(x,z) \leq d(x,y) + d(y,z)$, for all x,y,z in A.

FRAME 2

If $x = (x_1,x_2)$ and $y = (y_1,y_2)$ are points of $\mathbb{R}^2$ then

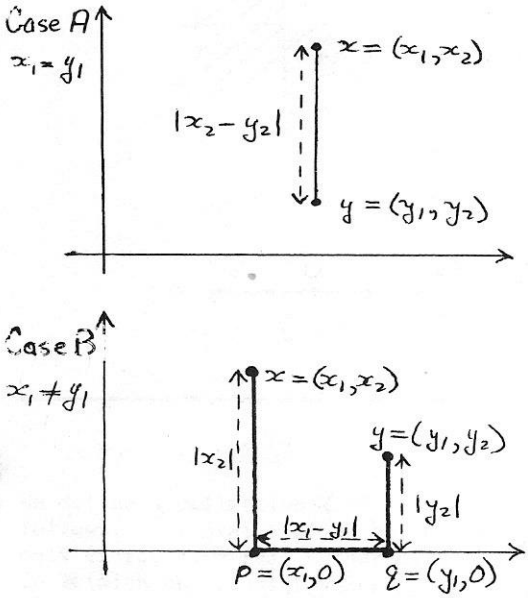
$$d(x,y) = \begin{cases} |x_2 - y_2| & \text{if } x_1 = y_1, \\ |x_2| + |y_2| + |x_1 - y_1| & \text{if } x_1 \neq y_1. \end{cases}$$

Case (A): if $x_1 = y_1$, then $d(x,y) = |x_2 - y_2|$.

Case (B): if $x_1 \neq y_1$, then

$$d(x,y) = |x_2| + |y_2| + |x_1 - y_1|.$$

FRAME 3



FRAME 4

(M1): $d(x,y) \geq 0$ for all x,y.

$d(x,x) = 0$ for all x.

Let $x = (x_1,x_2)$, $y = (y_1,y_2)$ be distinct points of $\mathbb{R}^2$.

If $x_1 = y_1$, then $x_2 \neq y_2$ and

$$d(x,y) = |x_2 - y_2| > 0.$$

If $x_1 \neq y_1$, then

$$d(x,y) = |x_2| + |y_2| + |x_1 - y_1| \geq |x_1 - y_1| > 0.$$

FRAME 5

(M3): Let $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $z = (z_1,z_2)$ be points of $\mathbb{R}^2$. There are five cases to consider.

(i) x_1, y_1 and z_1 are all different.

Then $d(x,z) = |x_2| + |z_2| + |x_1 - z_1|$

and

$d(x,y)+d(y,z) = |x_2|+2|y_2|+|z_2|+|x_1-y_1|+|y_1-z_1|$.

Since $|y_2| \geq 0$ and $|x_1-z_1| \leq |x_1-y_1| + |y_1-z_1|$ it follows that $d(x,z) \leq d(x,y) + d(x,z)$.

(ii) $x_1 = y_1 \neq z_1$.

Then $d(x,z) = |x_2| + |z_2| + |x_1-z_1|$

and

$d(x,y)+d(y,z) = |x_2-y_2| + |y_2| + |z_2| + |y_1-z_1|$.

Since $|x_1-z_1| = |y_1-z_1|$ and $|x_2| = |(x_2-y_2)+y_2| \leq |x_2-y_2|+|y_2|$, it follows that $d(x,z) \leq d(x,y) + d(y,z)$.

(iii) $x_1 \neq y_1 = z_1$.

Since (M2) is satisfied, the argument in this case is the same as case (ii).

(iv) $x_1 = z_1 \neq y_1$.

Then $d(x,z) = |x_2-z_2|$ and

$d(x,y)+d(y,z) = |x_2|+2|y_2|+|z_2|+|x_1-y_1|+|y_1-z_1|$.

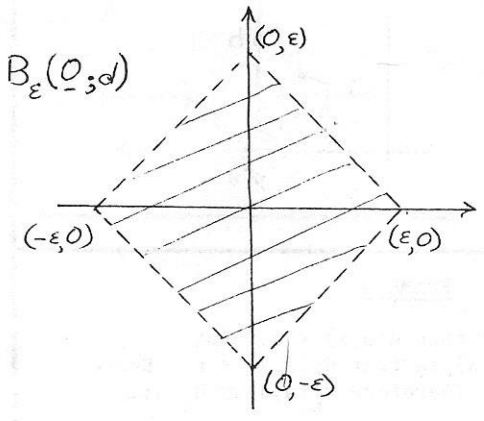
Since $|x_2-z_2| \leq |x_2| + |z_2|$ it follows that $d(x,z) \leq d(x,y) + d(y,z)$.

(v) $x_1 = y_1 = z_1$.

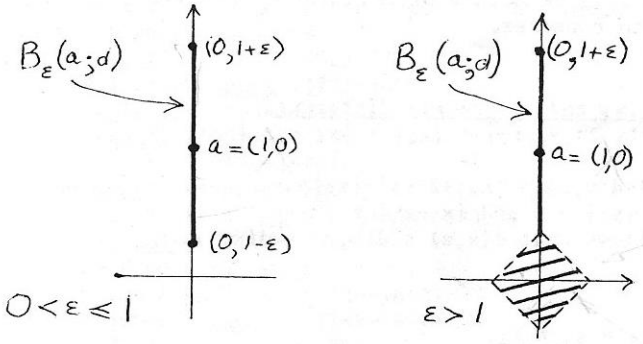
In this case

$d(x,z) = |x_2-z_2| \leq |x_2-y_2|+|y_2-z_2| \leq d(x,y)+d(y,z)$.

FRAME 6



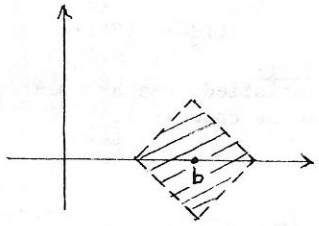
FRAME 7



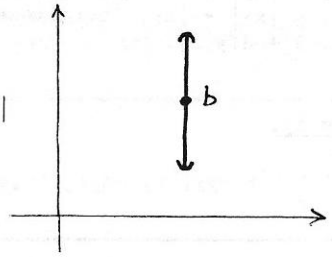
FRAME 8

$b=(b_1, b_2)$

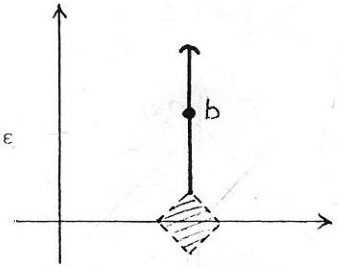
$B_\varepsilon(b;d)$, $b_2 = 0$



$B_\varepsilon(b;d)$, $0<\varepsilon\leq|b_2|$



$B_\varepsilon(b;d)$, $0<|b_2|<\varepsilon$



FRAME 9

If $x \in B_\varepsilon(a;d)$ then $d(a,x) < \varepsilon$. But $d_2(a,x) \leq d(a,x)$ so that $d_2(a,x) < \varepsilon$. Hence $x \in B_\varepsilon(a;d_2)$. Therefore $B_\varepsilon(a;d) \subset B_\varepsilon(a;d_2)$.

An Example of a Topological Space

(Unit 4) Tape 1 Side 2

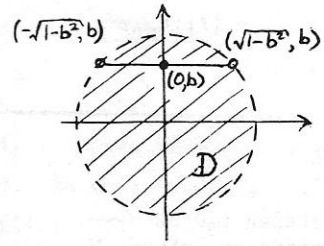
FRAME 1

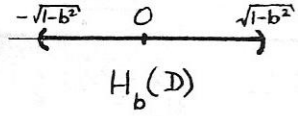
A topology for a non-empty set A consists of a collection $\mathcal{T}$ of subsets of A satisfying
(T1) A, $\emptyset \in \mathcal{T}$;
(T2) the intersection of two members of $\mathcal{T}$ is a member of $\mathcal{T}$;
(T3) the union of any collection of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

FRAME 2

If $V \subset \mathbb{R}^2$ and $b \in \mathbb{R}$ then
$H_b(V) = \{x \in \mathbb{R} : (x,b) \in V\}$.

FRAME 3





$H_b(D)$

FRAME 4

We define a collection $\mathcal{U}$ of subsets of $\mathbb{R}^2$ as follows. A subset V of $\mathbb{R}^2$ belongs to $\mathcal{U}$ if and only if for each real number b, $H_b(V)$ is open in $\mathbb{R}$ (with the usual topology).

FRAME 5

(T1): For every b,
$H_b(\mathbb{R}^2) = \mathbb{R}$, $H_b(\emptyset) = \emptyset$.

FRAME 6

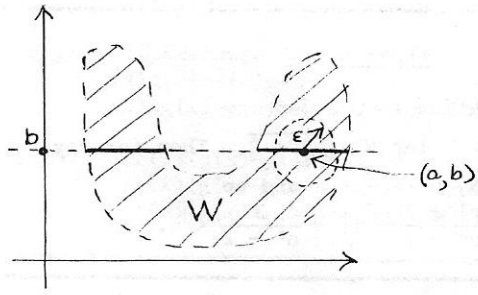
(T2): Let $V_1, V_2 \in \mathcal{U}$. For each real number b,
$x \in H_b(V_1 \cap V_2)$
$\iff (x,b) \in V_1 \cap V_2$
$\iff (x,b) \in V_1$ and $(x,b) \in V_2$
$\iff x \in H_b(V_1)$ and $x \in H_b(V_2)$
$\iff x \in H_b(V_1) \cap H_b(V_2)$.
Thus $H_b(V_1 \cap V_2) = H_b(V_1) \cap H_b(V_2)$.

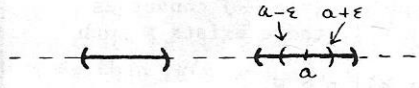
Since $H_b(V_1)$ and $H_b(V_2)$ are open in $\mathbb{R}$, it follows that $H_b(V_1 \cap V_2)$ is open in $\mathbb{R}$. This is true for every b, so $V_1 \cap V_2 \in \mathcal{U}$.

FRAME 7

(T3): Let $(V_i)_{i \in I}$ be a family of members of $\mathcal{U}$ and let $V = \bigcup_{i \in I} V_i$. For each real number b, $H_b(V) = \bigcup_{i \in I} H_b(V_i)$. Thus for each b, $H_b(V)$ is a union of sets open in $\mathbb{R}$ and so is open in $\mathbb{R}$. Hence $V \in \mathcal{U}$.

FRAME 8





$H_b(W)$

Sequences

(Unit 2) Tape 2 Side 1

FRAME 1

S Definition 1.2.2.
The sequence (s_n) converges to ℓ if given any $\varepsilon > 0$ there exists N such that $|s_n - \ell| < \varepsilon$ for all $n \geq N$.

FRAME 2

Problem: Does the sequence (s_n) given by $s_n = \frac{n^2 + 1}{n^2 - 2}$ converge?

$(s_n) = -2, \frac{5}{2}, \frac{10}{7}, \frac{17}{14}, \ldots, \frac{10001}{9998}, \ldots \to 1?$

$$|s_n - 1| = \left|\frac{n^2 + 1}{n^2 - 2} - 1\right| = \frac{3}{|n^2 - 2|}.$$

For $\varepsilon = \frac{1}{2}$,

$$|s_n - 1| < \varepsilon \iff \frac{3}{|n^2 - 2|} < \tfrac{1}{2} \iff |n^2 - 2| > 6.$$

Taking $N = 3$, $|s_n - 1| < \varepsilon$ for all $n \geq N$.

Exercise: Find N for $\varepsilon = \frac{1}{500}$.

FRAME 3

Solution to Exercise

$|s_n - 1| = \frac{3}{|n^2 - 2|} < \frac{1}{500}$ provided $|n^2 - 2| > 1500$. This is true for all $n \geq 39$, so take $N = 39$.

FRAME 4

Complete Solution to problem

Given any $\varepsilon > 0$ let $N > \sqrt{\frac{3}{\varepsilon} + 2}$. Then for any integer $n \geq N$, $n^2 > \frac{3}{\varepsilon} + 2$ and so

$$|s_n - 1| = \left|\frac{n^2 + 1}{n^2 - 2} - 1\right| = \frac{3}{n^2 - 2} < \varepsilon.$$

FRAME 5

Example of non-convergence

Suppose that the sequence $(\sin \frac{n\pi}{2})$ converges to ℓ. Then, taking $\varepsilon = \frac{1}{2}$, there exists N such that

$|\sin \frac{n\pi}{2} - \ell| < \frac{1}{2}$ for all $n \geq N$.

In particular, for $n = 4N + 1$ $(\sin \frac{n\pi}{2} = 1)$

$|1 - \ell| < \frac{1}{2} \Rightarrow \frac{1}{2} < \ell < \frac{3}{2}$

And for $n = 4N + 3$ $(\sin \frac{n\pi}{2} = -1)$

$|-1 - \ell| < \frac{1}{2} \Rightarrow -\frac{3}{2} < \ell < -\frac{1}{2}$.

Contradiction. No such limit ℓ exists.

FRAME 6

Problem: Sutherland Proposition 1.2.11 (b). Prove that if (s_n) and (t_n) converge respectively to ℓ and m then $(s_n t_n)$ converges to ℓm.

FRAME 7

Solution. What do we know?
Given $\varepsilon > 0$:

(i) (s_n) converges to ℓ means that there exists an integer N_1 such that $|s_n - \ell| < \varepsilon$ for all $n \geq N_1$.

(ii) Similarly, there exists N_2 such that $|t_n - m| < \varepsilon$ for all $n \geq N_2$.

FRAME 8

Solution continued. The connection —

$$\begin{aligned} |s_n t_n - \ell m| &= |s_n t_n - t_n \ell + t_n \ell - \ell m| \\ &= |t_n(s_n - \ell) + \ell(t_n - m)| \\ &\leq |t_n(s_n - \ell)| + |\ell(t_n - m)| \\ &= |t_n||s_n - \ell| + |\ell||t_n - m| \\ &[\leq \varepsilon\{|t_n| + |\ell|\}]. \end{aligned}$$

FRAME 9

Solution continued. Getting rid of n —
Since $(t_n) \to m$ there exists an integer N_3 such that $|t_n - m| < 1$ for all $n \geq N_3$.
Therefore $|t_n| = |(t_n - m) + m|$
$\leq |t_n - m| + |m| \leq 1 + |m|$
for all $n \geq N_3$.

FRAME 10

Solution completed. Tying the ends together —
Let $N = \max\{N_1, N_2, N_3\}$. Then for all $n \geq N$:

$|s_n - \ell| < \varepsilon$, $|t_n - m| < \varepsilon$ and $|t_n| < |m| + 1$.

Therefore

$|s_n t_n - \ell m| < \varepsilon\{|t_n| + |\ell|\} < \varepsilon\{|\ell| + |m| + 1\}$.

FRAME 11

A formal proof.
Proof. Given any $\varepsilon > 0$ there exist integers N_1, N_2 and N_3 such that

(i) $|s_n - \ell| < \varepsilon$ for all $n \geq N_1$,

(ii) $|t_n - m| < \varepsilon$ for all $n \geq N_2$, and

(iii) $|t_n - m| < 1$ for all $n \geq N_3$.

Let $N = \max\{N_1, N_2, N_3\}$. Then for $n \geq N$:
$|t_n| \leq |t_n - m| + |m| \leq 1 + |m|$;
and so

$$\begin{aligned} |s_n t_n - \ell m| &= |t_n(s_n - \ell) + \ell(t_n - m)| \\ &\leq |t_n||s_n - \ell| + |\ell||t_n - m| \\ &\leq (|m| + 1)\varepsilon + |\ell|\varepsilon \\ &\leq (|\ell| + |m| + 1)\varepsilon. \end{aligned}$$

Continuity

(Unit 2) Tape 2 Side 1 and 2

FRAME 1

S. Definition 1.4.3'.
The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if given any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for any x satisfying $|x - a| < \delta$.

FRAME 2

Problem. Prove that $f(x) = 2x^2 + x - 1$ is continuous at $x = 1$.

$$\begin{aligned} |f(x) - f(1)| &= |2x^2 + x - 1 - 2| \\ &= |2x^2 + x - 3| \\ &= |(2x + 3)(x - 1)| \\ &= |2x + 3||x - 1| \quad [< \varepsilon?] \end{aligned}$$

FRAME 3

Solution to problem.
Given any $\varepsilon > 0$ choose $\delta = \min\{1, \frac{1}{7}\varepsilon\}$.
If $|x - 1| < \delta$ then $|x - 1| < 1$ and so $x \in (0,2)$. In this case $|2x + 3| < 7$ and
$|f(x) - f(1)| = |2x + 3||x - 1| < 7\delta \leq 7.\frac{1}{7}\varepsilon = \varepsilon.$
Exercise: Prove that f is continuous at $x = -1$.

FRAME 4

Solution to Exercise.
$$\begin{aligned} |f(x) - f(-1)| &= |2x^2 + x - 1 - 0| \\ &= |2x - 1||x + 1|. \end{aligned}$$
Given any $\varepsilon > 0$, let $\delta = \min\{1, \frac{1}{5}\varepsilon\}$. If $|x + 1| < \delta$, then $|x + 1| < 1$ so that $-2 < x < 0$ and $|2x - 1| < 5$. Therefore
$|f(x) - f(-1)| < 5\delta \leq 5.\frac{1}{5}\varepsilon = \varepsilon.$

FRAME 5

Problem: Is f continuous everywhere?
$$\begin{aligned} |f(x) - f(a)| &= |2x^2 + x - 1 - (2a^2 + a - 1)| \\ &= |2(x^2 - a^2) + (x - a)| \\ &= |(x - a)(2x + 2a + 1)| \\ &= |x - a||2x + 2a + 1|. \end{aligned}$$
At $x = a - 1$, $2x + 2a + 1 = 4a - 1$.
At $x = a + 1$, $2x + 2a + 1 = 4a + 3$.
Therefore on $(a-1, a+1)$,
$$|2x+2a+1| < \max\{|4a-1|, |4a+3|\}.$$

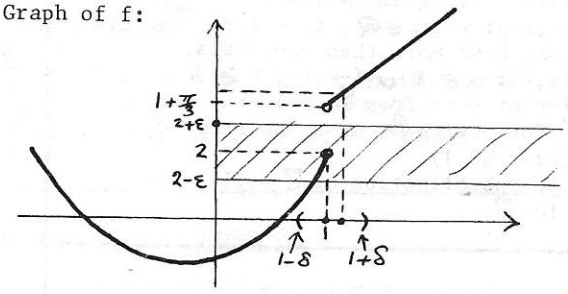
FRAME 6

Solution to problem
Given any $\varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{k}\}$ where $k = \max\{|4a-1|, |4a+3|\}$. If $|x - a| < \delta$ then $|x - a| < 1$ and so $a - 1 < x < a + 1$. In this case $|2x + 2a + 1| < k$ and
$$\begin{aligned} |f(x) - f(a)| &= |2x + 2a + 1||x - a| \\ &< k\delta \leq k.\frac{\varepsilon}{k} = \varepsilon. \end{aligned}$$

FRAME 7

Example of a discontinuous function.
$$f(x) = \begin{cases} 2x^2 + x - 1, & \text{if } x \leq 1, \\ x + \frac{\pi}{3}, & \text{if } x > 1. \end{cases}$$

Graph of f:



FRAME 8

Proof of discontinuity of f at x = 1.
Choose $\varepsilon = .01$. For any $\delta > 0$, $x = 1 + \frac{\delta}{2}$ satisfies $|x - 1| < \delta$. However
$$\begin{aligned} |f(1 + \tfrac{\delta}{2}) - f(1)| &= |1 + \tfrac{\delta}{2} + \tfrac{\pi}{3} - 2| \\ &= \tfrac{\pi}{3} - 1 + \tfrac{\delta}{2} > \tfrac{\pi}{3} - 1 > \varepsilon. \end{aligned}$$

FRAME 9

Problem: S page 17, Exercise 19.
$$\begin{aligned} fg(x)-fg(a) &= f(x)g(x)-f(a)g(a) \\ &= f(x)(g(x)-g(a))+g(a)(f(x)-f(a)). \end{aligned}$$
Therefore
$$\begin{aligned} |fg(x)-fg(a)| &= |f(x)(g(x)-g(a))+g(a)(f(x)-f(a))| \\ &\leq |f(x)(g(x)-g(a))|+|g(a)(f(x)-f(a))| \\ &= |f(x)||g(x)-g(a)|+|g(a)||f(x)-f(a)|. \end{aligned}$$

FRAME 10

Solution. What do we know?
Given any $\varepsilon > 0$.
As f is continuous at a there exists $\delta_1 > 0$ such that if $|x - a| < \delta_1$ then $|f(x) - f(a)| < \varepsilon$.
Similarly, there exists $\delta_2 > 0$ such that if $|x - a| < \delta_2$ then $|g(x) - g(a)| < \varepsilon$.
Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x - a| < \delta$,
$$\begin{aligned} |fg(x)-fg(a)| &\leq |f(x)||g(x)-g(a)|+|g(a)||f(x)-f(a)| \\ &< |f(x)|\,\varepsilon + |g(a)|\,\varepsilon \\ &< \varepsilon\,\{|f(x)| + |g(a)|\}. \end{aligned}$$

FRAME 11

Solution completed.
Since f is continuous at a there exists $\delta_3 > 0$ such that if $|x - a| < \delta_3$, then $|f(x) - f(a)| < 1$.
In this case
$$\begin{aligned} |f(x)| &= |f(x) - f(a) + f(a)| \\ &\leq |f(x) - f(a)| + |f(a)| \leq 1 + |f(a)|. \end{aligned}$$
Let $\delta_4 = \min\{\delta, \delta_3\}$. Then if $|x - a| < \delta_4$,
$$\begin{aligned} |fg(x) - fg(a)| &< \varepsilon\,\{|f(x)| + |g(a)|\} \\ &< \varepsilon\,\{|f(a)| + |g(a)| + 1\}. \end{aligned}$$

Bases and the Product Topology

(Unit 4) Tape 2 Side 2

FRAME 1

[a] A subcollection $\mathcal{B} \subset \mathcal{T}$ is a basis for $\mathcal{T}$ if every set in $\mathcal{T}$ is a union of sets from $\mathcal{B}$. (S Definition 3.2.1.).

[b] $\mathcal{B}_1 = \{(a,b): a,b \in \mathbb{R},\ a < b\} - \mathcal{T}_1$.
$\mathcal{B}_2 = \{(r_1,r_2): r_1,r_2 \in \mathbb{Q},\ r_1 < r_2\} - \mathcal{T}_2$.
A topology may have more than one basis.

[c] $\mathcal{B}_3 = \{(a,\infty): a \in \mathbb{R}\} \cup \{(-\infty,b): b \in \mathbb{R}\}$.
$(0,1) \neq$ union of sets from $\mathcal{B}_3$. Is there a topology $\mathcal{T}'$ for $\mathbb{R}$ with $\mathcal{B}_3$ as a basis for $\mathcal{T}'$?
$(-\infty,1) \cap (0,\infty) = (0,1)$.

[d] Not every subcollection of $\mathcal{T}$ will serve as a basis for $\mathcal{T}$.

FRAME 2

[a] A collection $\mathcal{B}$ of subsets of A is a synthetic basis for A if
(B1) A is a union of sets from $\mathcal{B}$;
(B2) $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cap B_2$ is a union of sets from $\mathcal{B}$.

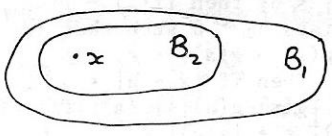
[b] Given a set A and a synthetic basis $\mathcal{B}$ on A, there is a corresponding topology $\mathcal{T}$ on A generated by $\mathcal{B}$ (S Proposition 3.2.3):
$\mathcal{T} = \{U \in 2^A: U \text{ is a union of sets from } \mathcal{B}\}$.
c.f. S Proposition 3.2.4 and Note (iv) of Section 4.3 of Unit.

[c] $\mathcal{B}_4 = \{(a,b) \cap \mathbb{Q}: a,b \in \mathbb{R}\} \cup \{(a,b) \cap (\mathbb{R}-\mathbb{Q}): a,b \in \mathbb{R}\} - \mathcal{T}_4$.
Usual topology on $\mathbb{R}$, $\mathcal{T}_1 \subset \mathcal{T}_4$; for
$U = \bigcup (a_i,b_i)$
$= \bigcup [\{(a_i,b_i) \cap \mathbb{Q}\} \cup \{(a_i,b_i) \cap (\mathbb{R}-\mathbb{Q})\}]$.

[d] $(0,1) \cap \mathbb{Q} \in \mathcal{T}_4, \notin \mathcal{T}_1$.

FRAME 3

[a] PROPOSITION. Let $\mathcal{B}_1, \mathcal{B}_2$ be bases for the topological spaces $\{A,\mathcal{T}_1\}$, $\{A,\mathcal{T}_2\}$. Then
$\mathcal{T}_1 \subset \mathcal{T}_2 \iff$ (*) for each $B_1 \in \mathcal{B}_1$ and each $x \in B_1$, there is some $B_2 \in \mathcal{B}_2$ with $x \in B_2 \subset B_1$.



[b] Assume $\mathcal{T}_1 \subset \mathcal{T}_2$.
$B_1 \in \mathcal{B}_1 \Rightarrow B_1 \in \mathcal{T}_1 \subset \mathcal{T}_2$
$\Rightarrow B_1 =$ union of sets in $\mathcal{B}_2$.
$x \in B_1 \Rightarrow x \in$ some B_2 in the collection; $B_2 \subset B_1$
$\Rightarrow x \in B_2 \subset B_1$.

Assume (*).
$U \in \mathcal{T}_1 \Rightarrow U =$ union of sets (e.g. B_1) in $\mathcal{B}_1$.
Corresponding to each x in B_1 choose some $B_2^{(x)} (\subset B_1)$ in $\mathcal{B}_2$, using (*).
$V =$ union of all such sets $B_2^{(x)}$.
Then $U \subset \bigcup B_1 \subset \bigcup \{\bigcup_{x \in B_1} B_2^{(x)}\} = V$;
$B_2^{(x)} \subset B_1 \Rightarrow V \subset \bigcup \{B_2^{(x)}\} \subset \bigcup B_1 = U$.

(Continued in next column)

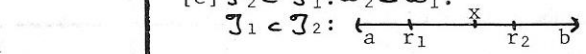
FRAME 3 Continued

[c] PROPOSITION (ctd.) $\mathcal{T}_2 \subset \mathcal{T}_1$
$\iff$ (**) .....
.....

$\mathcal{T}_1 = \mathcal{T}_2$
$\iff$ .....
.....

[d] Usual topology on $\mathbb{R}$, $\mathcal{T}_1 \subset \mathcal{T}_4$.
$x \in (a,b) \cap \mathbb{Q}$ or $(a,b) \cap (\mathbb{R}-\mathbb{Q}) \subset (a,b)$

[e] $\mathcal{T}_2 \subset \mathcal{T}_1$: $\mathcal{B}_2 \subset \mathcal{B}_1$.
$\mathcal{T}_1 \not\subset \mathcal{T}_2$:



FRAME 4

[a] $A = \mathbb{R}$
$f : x \mapsto x^2$, $g : x \mapsto x \mapsto x^3$ are continuous.

[b]
$$f^{-1}((a,b)) = \begin{cases} \varnothing & \text{if } b \leq 0, \\ (-b^{\frac{1}{2}}, b^{\frac{1}{2}}) & \text{if } a < 0 < b, \\ (a^{\frac{1}{2}}, b^{\frac{1}{2}}) & \text{if } a \geq 0. \end{cases}$$

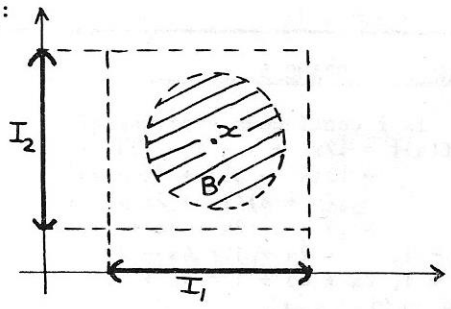
[c] $g^{-1}((a,b)) = (a^{1/3}, b^{1/3})$.

FRAME 5

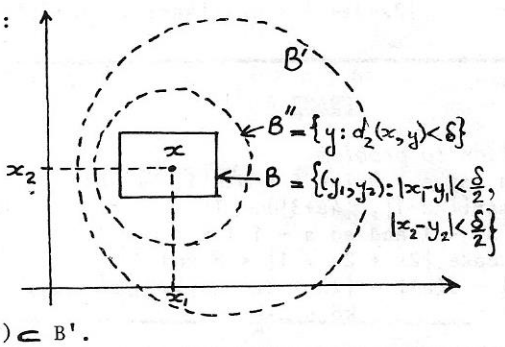
[a] $T_1 = \{A_1, \mathcal{T}_1\}$, $T_2 = \{A_2, \mathcal{T}_2\}$. The product topology $\mathcal{T}$ for $A_1 \times A_2$ is the topology with basis $\mathcal{B} = \{U_1 \times U_2: U_1 \in \mathcal{T}_1,\ U_2 \in \mathcal{T}_2\}$.

[b] $\mathcal{T}$ = product topology for $\mathbb{R} \times \mathbb{R}$
$\mathcal{T}'$ = 'usual' topology for $\mathbb{R}^2$
- basis $\mathcal{B}'$ of all open balls in $\mathbb{R}^2$.

[c] $\mathcal{T} \subset \mathcal{T}'$:



[d] $\mathcal{T}' \subset \mathcal{T}$:



$x \in B\ (\subset B'') \subset B'$.

Continuity — An Overall View

(Unit 4) Tape 3 Side 1

FRAME 1

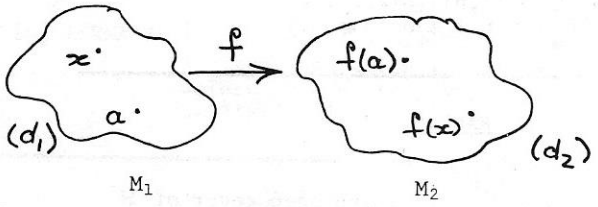
A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at the point $a \in \mathbb{R}$ if, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for all x with $|x - a| < \delta$. (Definition 1.4.3')
We say that f is continuous if f is continuous at all points $a \in \mathbb{R}$.

FRAME 2

[a] Metric spaces $M = \{A,d\}$.
Metric function $d : A \times A \to \mathbb{R}$ satisfies
(M1) $d(x,y) \geq 0$; $d(x,y) = 0 \iff x = y$;
(M2) $d(x,y) = d(y,x)$;
(M3) $d(x,y) + d(y,z) \geq d(x,z)$;
for all x, y, z in A.

[b] $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(f(x), f(a)) < \varepsilon$ for all x with $d(x,a) < \delta$.

[c] $M_1 = \{A_1,d_1\}$, $M_2 = \{A_2,d_2\}$ metric spaces.
S 2.1.3: A function $f : M_1 \to M_2$ is continuous at $a \in M_1$ if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that
$d_2(f(x),f(a)) < \varepsilon$ for all x with $d_1(x,a) < \delta$.



FRAME 3

Correspondence Principle: when generalizing a definition, the new definition must coincide with the old definition in the old setting.

FRAME 4

A topological space $T = \{A,\mathcal{T}\}$ is a set A and a collection $\mathcal{T}$ of subsets of A satisfying:
(T1) ϕ, $A \in \mathcal{T}$;
(T2) the intersection of any two sets in $\mathcal{T}$ is an element of $\mathcal{T}$;
(T3) the union of any collection of sets in $\mathcal{T}$ is an element of $\mathcal{T}$.

FRAME 5

[a] An open ball $B_\varepsilon(a;d)$ of $M = \{A,d\}$ is the set $B_\varepsilon(a;d) = \{x \in A: d(x,a) < \varepsilon\}$.

[b] $V \subset A$ is open in M if for each $a \in V$ there exists an ε, depending (in general) on a, such that $B_\varepsilon(a;d) \subset V$.

[c] S 2.3.6: $f : M_1 \to M_2$ is continuous at $a \in M_1$ if given any $B_\varepsilon(f(a))$ there exists a $B_\delta(a)$ such that $f(B_\delta(a)) \subset B_\varepsilon(f(a))$.

FRAME 6

[a] $f : A \to B$ with $C \subset A$, $D \subset B$:
$f(C) \subset D \iff C \subset f^{-1}(D)$.

[b] $f : M_1 \to M_2$ is continuous at $a \in M_1$ if given any $B_\varepsilon(f(a),d_2) \subset M_2$ there exists a $B_\delta(a,d_1) \subset M_1$ such that
$B_\delta(a,d_1) \subset f^{-1}(B_\varepsilon(f(a),d_2))$.

[c] $f : M_1 \to M_2$ is continuous if given any open set $V \subset M_2$, the inverse image $f^{-1}(V)$ is open in M. [S 2.3.13].

FRAME 7

A function $f : T_1 \to T_2$ is continuous if for every $V \in \mathcal{T}_2$, $f^{-1}(V) \in \mathcal{T}_1$.

FRAME 8

[a] The left-ray topology $\mathcal{T}$ on $\mathbb{R}$ is the collection ϕ, $\mathbb{R}$ and $(-\infty,a)$ for all $a \in \mathbb{R}$; $T = \{\mathbb{R},\mathcal{T}\}$.

[b] For the identity function $i : T \to T$, $i(x) = x$, we have $i^{-1}(V) = V$ for any subset $V \subset \mathbb{R}$.

[c] For $f : T \to T$, $f(x) = x^2$, we have $f^{-1}(\phi) = \phi$, $f^{-1}(\mathbb{R}) = \mathbb{R}$, and

$$f^{-1}((-\infty,a)) = \begin{cases} (-a^{\frac{1}{2}},a^{\frac{1}{2}}) & \text{if } a > 0, \\ \phi & \text{if } a \leq 0. \end{cases}$$

[d] For $g : T \to T$, $g(x) = x^3$, we have $g^{-1}(\phi) = \phi$, $g^{-1}(\mathbb{R}) = \mathbb{R}$, $g^{-1}((-\infty,a)) = (-\infty,a^{1/3})$.

Compactness

(Unit 6) Tape 3 Side 1

FRAME 1

Definition of a compact space

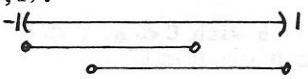
Let $\{X,\mathcal{T}\}$ be a topological space.

1. An open cover of $\{X,\mathcal{T}\}$ is a collection of open sets whose union is the whole of X.
2. A subcover of an open cover is a sub-collection of the open cover which is itself a cover.
3. $\{X,\mathcal{T}\}$ is compact if every open cover of it has a finite subcover.

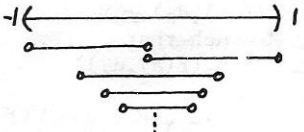
FRAME 2

Open covers of $I = (-1,1)$.

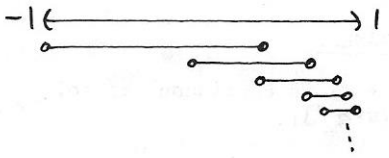
1. $\{(-1,\frac{1}{2}),\ (-\frac{1}{2},1)\}$.



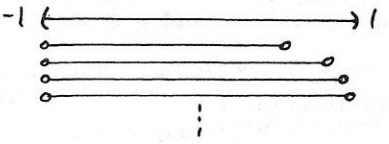
2. $\{(-1,0),\ (0,1),\ (-\frac{1}{2},\frac{1}{2}),\ (-\frac{1}{3},\frac{1}{3}),\ \ldots$
$\ldots,\ (-\frac{1}{n},\frac{1}{n}),\ \ldots\}$.



3. $\{(-1,\frac{1}{2}),\ (0,\frac{2}{3}),\ (\frac{1}{2},\frac{3}{4}),\ (\frac{2}{3},\frac{4}{5}),\ (\frac{3}{4},\frac{5}{6}),$
$\ldots,\ (\frac{n-1}{n},\frac{n+1}{n+2}),\ \ldots\}$.



4. $\{(-1,\frac{1}{2}),\ (-1,\frac{2}{3}),\ (-1,\frac{3}{4}),\ \ldots,\ (-1,\frac{n}{n+1}),\ldots\}$.



FRAME 3

Example 4 has no finite subcover

Suppose $\{(-1,\frac{n_1}{n_1+1}),\ (-1,\frac{n_2}{n_2+1}),\ \ldots,\ (-1,\frac{n_r}{n_r+1})\}$ is a cover of I.

$\bigcup_{i=1}^{r}(-1,\frac{n_i}{n_i+1}) = (-1,\frac{N}{N+1}) \not\supset (-1,1)$,

where $N = \max\{n_1,n_2,\ldots,n_r\}$. Contradiction.

FRAME 4

An infinite open cover of $J = [-1,1]$.

$\{[-1,-1+\varepsilon),\ (1-\delta,1],\ (-1,\frac{1}{2}),(0,\frac{2}{3}),(\frac{1}{2},\frac{3}{4}),\ldots$

$\ldots,(\frac{n-1}{n},\frac{n+1}{n+2})\ldots\}$

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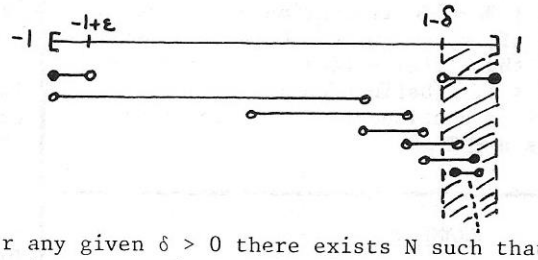
FRAME 4 Continued



For any given $\delta > 0$ there exists N such that $\delta > \frac{1}{N}$. Then $\frac{n-1}{n} = 1 - \frac{1}{n} > 1 - \delta$ for all $n \geq N$.

Therefore $(\frac{n-1}{n},\frac{n+1}{n+2}) \subset (1-\delta,\ 1]$ for all $n \geq N$,

and so $\{[-1,-1+\varepsilon),(1-\delta,1],(-1,\frac{1}{2}),(0,\frac{2}{3}),(\frac{1}{2},\frac{3}{4}),\ldots$

$\ldots,(\frac{N-2}{N-1},\frac{N}{N+1})\}$ is a finite subcover.

FRAME 5

Problems: Is $\{\mathbb{N},\mathcal{T}_i\}$ compact, where

$\mathbb{N} = \{1,2,3,4,5,\ldots\}$,

$\mathcal{T}_1$ = discrete topology,

$\mathcal{T}_2 = \{U \in \mathcal{T}_2 \iff U = \phi$ or $\mathbb{N}-U$ is finite$\}$,

$\mathcal{T}_3 = \{\phi,O_1,O_2,O_3,\ldots,O_n,\ldots\}$
where $O_n = \{n,n+1,n+2,\ldots\}$,

$\mathcal{T}_4 = \{U \in \mathcal{T}_4 \iff U = \phi$ or $3 \in U\}$?

FRAME 6

$\{\mathbb{N},\mathcal{T}_1\}$ is not compact.

Proof. $\{\{1\},\{2\},\{3\},\ldots\}$ is an open cover of $\mathbb{N}$ which has no finite subcover.

FRAME 7

$\{\mathbb{N},\mathcal{T}_2\}$ is compact.

Proof. Let $\mathcal{U}$ be any open cover of $\mathbb{N}$. Let $U_0 \neq \phi$ be any member of $\mathcal{U}$. By definition of $\mathcal{T}_2$, $\mathbb{N} - U_0$ is finite, say $\mathbb{N} - U_0 = \{n_1,n_2,\ldots,n_k\}$. Now for each $i = 1,2,\ldots,k, n_i \in U_i$ for some $U_i \in \mathcal{U}$. Therefore $\{U_0,U_1,\ldots,U_k\}$ is a finite subcover of $\mathcal{U}$.

FRAME 8

$\{\mathbb{N},\mathcal{T}_3\}$ is compact.

Proof. Let $\mathcal{U}$ be any open cover of $\mathbb{N}$. Then $1 \in U$ for some $U \in \mathcal{U}$. But the only open subset of $\mathbb{N}$ which contains 1 is $O_1 = \mathbb{N}$. Therefore $U = \mathbb{N} \in \mathcal{U}$ and $\{\mathbb{N}\}$ is a finite subcover of $\mathcal{U}$.

FRAME 9

$\{\mathbb{N},\mathcal{T}_4\}$ is not compact.

Proof. $\{\{3,1\},\ \{3,2\},\ \{3,4\},\ \{3,5\},\ \{3,6\},$ $\ldots,\ \{3,n\},\ \ldots\}$ is an open cover of $\mathbb{N}$ which has no finite subcover.

The Cantor Set

(Unit 6) Tape 3 Side 2

FRAME 1

Representations of numbers in [0,1]

[a] DECIMAL: $x = .d_1d_2d_3\ldots \quad (d_i = 0,1,2,\ldots,9)$
$= \frac{d_1}{10^1} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \ldots$

[b] BINARY: $x = .b_1b_2b_3\ldots \quad (b_i = 0,1)$
$= \frac{b_1}{2^1} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \ldots$

[c] TERNARY: $x = .t_1t_2t_3\ldots \quad (t_i = 0,1,2)$
$= \frac{t_1}{3^1} + \frac{t_2}{3^2} + \frac{t_3}{3^3} + \ldots$

FRAME 2

Uniqueness of the representation

[a] DECIMAL: $\frac{1}{2} = 0.5000\ldots = 0.4999\ldots$
The representation is unique unless it ends in a string of 0's or 9's.

[b] BINARY: $\frac{1}{2} = 0.1000\ldots = 0.01111\ldots$
The representation is unique unless it ends in a string of 0's or 1's.

[c] TERNARY: $\frac{1}{3} = 0.1000\ldots = 0.0222\ldots$
The representation is unique unless it ends in a string of 0's or 2's.

FRAME 3

Strategy for construction of the Cantor Set K.
For each n, determine a set K_n.
Then $K = \bigcap_{n=1}^{\infty} K_n$.

FRAME 4

[a] Let $x \in [0,1]$ have ternary representation
$x = .t_1t_2t_3\ldots \quad (t_i = 0,1,2)$.

Delete all x whose representation necessarily has $t_1 = 1$.

[b] Case 1: no representation of x has $t_1 = 1$.

Case 2: x has two ternary representations, one of which has $t_1 = 1$, one has $t_1 \neq 1$.

Case 3: all representations of x have $t_1 = 1$.

(Continued in next column)

FRAME 4 Continued

[c] $\frac{3}{4} = .202020\ldots$
$\frac{5}{12} = .1020202\ldots$
$\frac{1}{3} = .1000\ldots = .0222\ldots$
$\frac{2}{3} = .2000\ldots = .1222\ldots$
$0 = .0000\ldots, \ 1 = .2222\ldots$

[d] $x \in (\frac{1}{3},\frac{2}{3}) \Rightarrow t_1 = 1$.
e.g. $\frac{5}{9} = .112222\ldots = .120000\ldots$

[e] $K_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1] \ (= [0,1] - (\frac{1}{3},\frac{2}{3}))$.

FRAME 5

Stage 2
[a] Starting with K_1 delete all x whose representation necessarily has $t_2 = 1$.

[b] $\frac{1}{9} = .01000\ldots = .00222\ldots$
$\frac{2}{9} = .02000\ldots = .01222\ldots$
$x \in (\frac{1}{9},\frac{2}{9}) \Rightarrow t_2 = 1$.

[c] $\frac{7}{9} = .21000\ldots = .20222\ldots$
$\frac{8}{9} = .22000\ldots = .21222\ldots$
$x \in (\frac{7}{9},\frac{8}{9}) \Rightarrow t_2 = 1$.

[d] $K_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$.

FRAME 6

Stage n
From the existing set K_{n-1} delete all x whose representation necessarily has $t_n = 1$.

FRAME 7

K is uncountable
[a] Each point of K has a unique ternary representation in which every digit is a 0 or a 2. Every string of 0's and 2's is the ternary representation of some point of K.

[b] $f : K \to [0,1]$ - obtained by replacing 2's by 1's in the ternary expansion.

[c] f is ONTO.
$\frac{1}{2} = f(\frac{1}{3}) = f(\frac{2}{3})$.

[d] (Proposition 12, Unit 1)
If A countably infinite, B infinite, $f : A \to B$ onto, then B is countably infinite.

[e] [0,1] uncountable (assumed without proof) $\Rightarrow$ K uncountable.

FRAME 8

$\frac{1}{4} \left(= \sum_{n=1}^{\infty} \frac{2}{3^{2n}}\right) = .020202\ldots \in K$.

The Topologists' Sine Curve

(Unit 7) Tape 3 Side 2

FRAME 1

For $x > 0$,

$$\sin\frac{1}{x} = 0 \iff \frac{1}{x} = n\pi,\ n \in \mathbb{N}$$
$$\iff x = \frac{1}{n\pi},\ n \in \mathbb{N}.$$

$$\sin\frac{1}{x} = 1 \iff \frac{1}{x} = \frac{\pi}{2} + 2n\pi,\ n \in \mathbb{N}$$
$$\iff x = \frac{2}{(4n+1)\pi},\ n \in \mathbb{N}.$$

$$\sin\frac{1}{x} = -1 \iff \frac{1}{x} = \frac{3\pi}{2} + 2n\pi,\ n \in \mathbb{N}$$
$$\iff x = \frac{2}{(4n+3)\pi},\ n \in \mathbb{N}.$$

FRAME 2

$f : I \to G \cup J$ is a path, and
$f(t) = (f_1(t), f_2(t))$, $t \in I = [0,1]$.

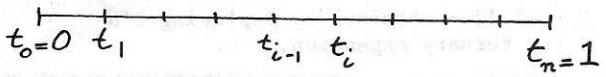
If $f_1(t) > 0$ then $f(t) \in G$, so that
$f_2(t) = \sin\frac{1}{f_1(t)}$

FRAME 3

Since $f_2 : I \to \mathbb{R}$ is uniformly continuous, for each positive real number ε there exists a positive real number δ with the property that $|f_2(t) - f_2(t')| < \varepsilon$ for all t, t' in I such that $|t - t'| < \delta$.

FRAME 4

Choose an integer n such that $n > \frac{1}{\delta}$. Put $t_i = \frac{i}{n}$, $i = 0,\ldots,n$. Then $t_0 = 0$, $t_n = 1$, and $t_i - t_{i-1} = \frac{1}{n} < \delta$ for $i = 1,\ldots,n$.



FRAME 5

For $i = 0,\ldots,n$, $\quad f_1(t_i) > \frac{2}{(2i+3)\pi}$.

FRAME 6

When we have proved the result stated in Frame 5 we know in particular that

$f_1(1) = f_1(t_n) > \frac{2}{(2n+3)\pi} > 0$.

But $f(1) = (f_1(1), f_2(1))$, so $f(1) \neq (0,0)$.

FRAME 7

Start of the induction: $f(0) = (\frac{1}{\pi}, 0)$, so that

$f_1(t_0) = f_1(0) = \frac{1}{\pi} > \frac{2}{3\pi}$.

FRAME 8

For the induction step we must prove that

$f_1(t_{i+1}) > \frac{2}{(2(i+1)+3)\pi} = \frac{2}{(2i+5)\pi}$.

FRAME 9

We have $f_1(t_i) > 2/(2i+3)\pi$ for some $i \geq 0$; and we assume that $f_1(t_{i+1}) \leq 2/(2i+5)\pi$. Then

$f_1(t_i) > \frac{2}{(2i+3)\pi} > \frac{2}{(2i+5)\pi} \geq f_1(t_{i+1})$.

We can now apply the I.V.P. of f_1 restricted to $[t_i, t_{i+1}]$, to assert that there exist t, t' in $[t_i, t_{i+1}]$ such that

$f_1(t) = \frac{2}{(2i+3)\pi}$ and $f_1(t') = \frac{2}{(2i+5)\pi}$.

Then

$$f_2(t) = \sin\frac{1}{f_1(t)} = \sin\left(i\pi + \frac{3\pi}{2}\right)$$

and

$$f_2(t') = \sin\frac{1}{f_1(t')} = \sin\left(i\pi + \frac{5\pi}{2}\right).$$

Therefore $|f_2(t)| = |f_2(t')| = 1$; and $f_2(t)$, $f_2(t')$ have opposite signs so that $|f_2(t) - f_2(t')| = 2$. Since $|t - t'| \leq t_{i+1} - t_i < \delta$, this contradicts the definition of δ.

Hence

$f_1(t_{i+1}) > \frac{2}{(2i+5)\pi}$, as required.